Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes

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We consider the thermodynamics of rotating and charged asymptotically de Sitter black holes. Using Hamiltonian perturbation theory techniques, we derive three different first law relations including variations in the cosmological constant, and associated Smarr formulas that are satisfied by such spacetimes. Each first law introduces a different thermodynamic volume conjugate to the cosmological constant. We examine the relation between these thermodynamic volumes and associated geometric volumes in a number of examples, including Kerr-dS black holes in all dimensions and Kerr-Newman-dS black holes in $D = 4$. We also show that the Chong-Cvetic-Lu-Pope solution of $D = 5$ minimal supergravity, analytically continued to positive cosmological constant, describes black hole solutions of the Einstein-Chern-Simons theory and include such charged asymptotically de Sitter black holes in our analysis. In all these examples we find that the particular thermodynamic volume associated with the region between the black hole and cosmological horizons is equal to the naive geometric volume. Isoperimetric inequalities, which hold in the examples considered, are formulated for the different thermodynamic volumes and conjectured to remain valid for all asymptotically de Sitter black holes. In particular, in all examples considered, we find that for fixed volume of the observable universe, the entropy is increased by adding black holes. We conjecture that this is true in general.

I. INTRODUCTION

That black holes have a temperature proportional to their surface gravity and obey a version of the first law of thermodynamics is a notion that has been of continued interest for over three decades. Studies of black holes that are asymptotically flat or anti de Sitter (AdS) have been carried out in considerable detail, with many corroborative results indicating that such objects indeed behave as thermodynamic systems. Thermodynamic equilibrium is straightforward to define, and (depending on the kind of black hole and its asymptotics) a variety of interesting phenomena emerge, such as large/small AdS black hole phase transitions [1]. An interesting new development in the study of black hole thermodynamics was the proposal that the mass of an AdS black hole should be interpreted as the enthalpy of the spacetime. This notion emerges from geometric derivations of the Smarr formula for AdS black holes which suggest that the cosmological constant should be considered as a thermodynamic variable analogous to pressure in the first law [2–17]. Further, this notion led to a *reverse* isoperimetric inequality conjecture [8] for AdS black holes, which says that for fixed thermodynamic volume, the entropy of an AdS black hole is maximized for Schwarzchild AdS. Since black holes in a spacetime with positive Λ are of interest in cosmology, in this paper we study these issues for de Sitter black holes.

Studies of asymptotically de Sitter (dS) black holes are somewhat more sparse, and need to confront two in-

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teresting and related issues. It is fundamental to the first law for asymptotically flat black holes that there is a Killing field which is timelike everywhere outside the black hole and hence can be used to define the mass. Adding a multiple of the rotation Killing field to this timelike Killing field gives the generator of the horizon, and the Killing generator defines the temperature. The same is true with negative cosmological constant. However in de Sitter there is no Killing field that is timelike everywhere outside the black hole, including infinity. Although generalizations of the first law in de Sitter were considered quite some time ago [18], a further key issue is that an observer in any static patch of the spacetime is between two horizons, each with its own distinct constant surface gravity. It is consequently unclear how to understand the thermodynamics of such spacetimes, since there is neither thermodynamic equilibrium nor an asymptotic region where a conserved mass-energy can be defined[1]. Various approaches have been taken toward addressing this problem, yielding mixed results.

In this paper we consider the thermodynamics of de Sitter black holes, focusing on the cosmological constant as a thermodynamic variable together with its conjugate potential. To highlight the correspondence with classical thermodynamics we also work with the pressure P related to the cosmological constant Λ according to

$$P = -\frac{\Lambda}{8\pi}, \qquad (1.1)$$

as it would be in a perfect fluid stress-energy. The potential conjugate to P is a volume V, which we will call the thermodynamic volume as in [4, 8].

It was found in [3] that the thermodynamic volume of a Schwarzschild-AdS black hole is equal to the geometric volume V' of the black hole interior, computed using the full D-dimensional volume element on a t equals constant slice. The relation between thermodynamic and geometric volumes was studied in [8] for a variety of charged and rotating AdS black holes. Equality between the two volumes was found to hold only in the static case (including charge), while with rotation the volumes differ by a simple correction term. It was also conjectured in [8] and verified for a variety of black hole spacetimes that the thermodynamic volume satisfies an inequality with respect to the black hole horizon area that is precisely the reverse of the isoperimetric inequality of Euclidean space.

In this paper we will carry out a similar exploratory study of thermodynamic and geometric volumes for asymptotically de Sitter black holes. An important new feature in this case is the need to distinguish between

several different thermodynamic volumes. The geometric derivation of the first law and Smarr formula for AdS black holes [3] takes place on a timelike hypersurface stretching outward from the black hole horizon to spatial infinity. For dS black holes, we will see that additional relations are obtained by alternatively considering the region stretching outwards to infinity from the black hole and cosmological horizons, or the region between the black hole and cosmological horizons. Each of these introduces a different thermodynamic volume. We denote the thermodynamic volumes associated in this way with the black hole and cosmological horizons by V_h and V_c, while reserving V for that associated with the region between the two horizons[2]. The different thermodynamic volumes may be thought of as arising from different thermodynamic ensembles, e.g., V_h arises from varying the cosmological constant with the black hole horizon area held fixed, while V_c is relevant if instead the cosmological horizon area is fixed.

Turning to specific examples, for Kerr-de Sitter black holes we find that V_h and V_c are equal to their geometric counterparts V_h' and V_c' only for static spacetimes, each differing by a simple expression when the black hole has angular momentum. These relations are the same as in AdS [8]. However, for the region between the two horizons the rotation-dependent differences cancel, so that $V = V'$ holds in the rotating case as well. In line with this result, we find that the thermodynamic volumes V_h and V_c satisfy reverse isoperimetric inequalities, while V satisfies a true isoperimetric inequality. These inequalities may be interpreted as bounds on combinations of the black hole and cosmological entropies at a fixed thermodynamic volume. In particular, in all examples considered, we find that for fixed volume of the observable universe, the entropy is increased by adding black holes. We conjecture that this is true in general.

Our results for charged rotating black holes are less general. In four dimensions, one has the Kerr-Newman-de Sitter spacetimes and in this case we find results that simply extend those of the zero charge case. In five dimensions we consider charged, rotating black holes of $D = 5$ Einstein-Chern-Simons theory with positive cosmological constant. We find that an exact solution for such black holes can be obtained from the gauged supergravity solution of Chong, Cvetic, Lu, and Pope [28], by analytically continuing the gauge coupling to obtain positive values of the cosmological constant. We compute the thermodynamic volumes V_h, V_c and V for these spacetimes and verify that V coincides with the geometric volume. While we are able to analytically verify the isoperi-

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[1] Some consideration has been given to defining a mass-energy-like quantity at future/past infinity in asymptotically dS spacetimes [19–21]. Positivity of a conformal mass defined in this context was shown in [29]

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[2] Similar results have been obtained by Cai [22, 23] and by Sekiwa [24] in more limited contexts (see also [25] and [26]). A study of the thermodynamics of asymptotically flat and AdS rotating black holes that included both inner and outer horizons, which has elements in common with our approach, was recently carried out [27].

metric inequality for a singly-rotating Chong-Cvetic-Lu-Pope black hole, because of the complexity of the expressions we have not been able to analytically complete the study of isoperimetric inequalities for the most general case. We have, however, strong indications from numerical analysis that such inequalities remain satisfied.

The outline of our paper is as follows. In the next section the three first laws and the corresponding Smarr formulae are derived using the Hamiltonian perturbation techniques. The thermodynamics of the Kerr-deSitter black holes in all dimensions is discussed in Sec. III while the isoperimetric inequalities involving the corresponding thermodynamic volumes are discussed in Sec. IV. The charged de Sitter black holes, their thermodynamics and isoperimetric inequalities are studied in Sec. V. Sec. VI is devoted to the summary.

II. FIRST LAW AND SMARR FORMULA

In this section we apply the Hamiltonian formalism of general relativity [30–32] to derive the first law relations for rotating dS black holes including variations in the cosmological constant. As noted above, we will actually find three different first law relations associated with different choices for the hypersurface and the Killing vector which enter the derivation. Two of these relations are linearly independent. Each version of the first law leads to an associated Smarr relation via an overall scaling.

A. Gauss' law for perturbations with $\delta\Lambda$

The essence of this Hamiltonian perturbation theory method is as follows. In Einstein gravity with cosmological constant, suppose one has a black hole solution with a Killing field. Now consider solutions that are perturbatively close to this background solution, but are not required to have the original Killing symmetry. The linearized Einstein constraint equations on a hypersurface can be expressed in the form of a Gauss' law (see [30]), relating a boundary integral at infinity to a boundary integral at the horizon. The physical meaning of this Gauss' law relation depends on the choice of Killing field, as well as on the choice of hypersurface. Taking the generator l^a of a Killing horizon, together with an appropriate choice of a spacelike hypersurface, yields the usual first law for variation of the mass [31] for asymptotically flat or AdS black hole spacetimes.

Assume we have a foliation of a spacetime by a family of hypersurfaces denoted by Σ and the unit timelike normal to the hypersurfaces n^a, $n \cdot n = -1$. The spacetime metric can then be written as

$$g_{ab} = s_{ab} - n_a n_b, \qquad (2.1)$$

where s_{ab} is the metric on the hypersurfaces Σ and satisfies $s_a{}^b n_b = 0$. As usual, the dynamical variables in the

Hamiltonian formalism are the metric s_{ab} and its canonically conjugate momentum $\pi^{ab} = -\sqrt{s}(K^{ab} - K s^{ab})$. Here $K_{ab} = s_a{}^c \nabla_c n_b$ is the extrinsic curvature of a hypersurface Σ and K stands for its trace, $K = K^a{}_a$. (Similarly later we define $\pi = \pi^a{}_a$ and $h = h^a{}_a$.) We consider Hamiltonian evolution along the vector field ξ^a, which can be decomposed into its components normal and tangential to Σ according to

$$\xi^a = F n^a + \beta^a, \qquad (2.2)$$

with $F = -\xi \cdot n$ denoting the lapse function and β^a the shift. The gravitational Hamiltonian which evolves the system along ξ^a is then given by $\mathcal{H} = FH + \beta^a H_a$ with

$$H \equiv -2G_{ab}n^a n^b = -R^{(D-1)} + \frac{1}{|s|}\Big(\frac{\pi^2}{D-2} - \pi^{ab}\pi_{ab}\Big),$$

$$H_b \equiv -2G_{ac}n^a s_b^c = -2\,D_a(|s|^{-\frac{1}{2}}\pi^{ab}). \qquad (2.3)$$

Here $R^{(D-1)}$ is the scalar curvature for the metric s_{ab} and D_a is the derivative operator on the hypersurface Σ. With a cosmological constant stress-energy $8\pi T_b^a = -\Lambda g_b^a$, the constraint equations become

$$H = -2\Lambda, \quad H_b = 0. \qquad (2.4)$$

Let g_{ab} be a solution to the Einstein equation with cosmological constant, and assume that ξ^a is a Killing vector of g_{ab}. Now let the metric $\tilde{g}_{ab} = g_{ab} + \delta g_{ab}$ be the linear approximation to another solution to the Einstein equations with cosmological constant $\Lambda + \delta\Lambda$. Denote the Hamiltonian data for the background metric by s_{ab}, π^{ab}, the corresponding perturbations to the data by $h_{ab} = \delta s_{ab}$ and $p^{ab} = \delta\pi^{ab}$, and the linearized Hamiltonian and momentum constraints by δH and δH_a.

It was shown in [30–32] that a particular linear combination of the perturbed constraints can be written as a total derivative, $F\delta H + \beta^a \delta H_a = D_a B^a$, where the vector B^a is given by

$$B^a[\xi] = F(D^a h - D_b h^{ab}) - h D^a F + h^{ab}D_b F$$
$$+ \frac{1}{\sqrt{|s|}}\beta^b\big(\pi^{cd}h_{cd}s^a{}_b - 2\pi^{ac}h_{bc} - 2p^a{}_b\big). \qquad (2.5)$$

On the other hand, since h_{ab}, p^{ab} solve the linearized constraint equations with the cosmological constant perturbed by $\delta\Lambda$ we also have $F\delta H + \beta^a \delta H_a = 2\delta\Lambda \xi^a n_a = -2\delta\Lambda F$ and therefore

$$D_a B^a = -2F\delta\Lambda. \qquad (2.6)$$

This has the form of a Gauss' law relation with source proportional to $F\delta\Lambda$. In [3, 33] it was shown that since ξ^a is a Killing vector, this source may also be written as a total derivative. We define the Killing potential ω^{ab} associated with ξ^a to be an antisymmetric tensor satisfying

$$\nabla_c \omega^{cb} = \xi^b. \qquad (2.7)$$

The lapse function can now be written as $F = -D_c(\omega^{cb}n_b)$ and the relation (2.6) becomes

$$D_a(B^a - 2\delta\Lambda\,\omega^{ab}n_b) = 0. \qquad (2.8)$$

Let $\hat{V}$ be a volume contained in Σ which has inner and outer boundaries $\partial \hat{V}_{in,out}$. Integrating the differential relation (2.8) over $\hat{V}$ gives

$$\int_{\partial \hat{V}_{out}} dar_c \left(B^c[\xi] - 2\delta\Lambda\omega^{cb}n_b \right)$$
$$= \int_{\partial \hat{V}_{in}} dar_c \left(B^c[\xi] - 2\delta\Lambda\omega^{cb}n_b \right) . \qquad (2.9)$$

Here we have let r_c denote the unit normal on each boundary, with the convention that r^c points into $\hat{V}$ on the inner boundary and out of $\hat{V}$ on the outer boundary, *i.e.* "towards infinity" on both boundaries. In the following we will consider different cases, in which these boundaries may be taken to be at the black hole horizon, the de Sitter horizon, and at infinity.

The different boundary integrals have important geometrical meanings. For an asymptotically flat or AdS black hole the variation in the ADM mass and angular momentum are given by the boundary integrals at infinity using the time translation $(\partial/\partial t)^a$ and rotation $(\partial/\partial\phi)^a$ Killing vectors, respectively. Also, the integral of the boundary term over the horizon using the horizon generating Killing vector is proportional to the surface gravity times the change in area of the black hole. Explicitly,

$$16\pi\delta\mathcal{M} = -\int_\infty dar_c B^c[\partial/\partial t] , \qquad (2.10)$$

$$16\pi\delta\mathcal{J} = \int_\infty dar_c B^c[\partial/\partial\phi] , \qquad (2.11)$$

$$2\kappa_{bh}\delta\mathcal{A}_h = -\int_h dar_c B^c[\partial/\partial t + \Omega_h\partial/\partial\phi] , \quad (2.12)$$

where $\mathcal{M}$ and $\mathcal{J}$ are the ADM mass and momentum, κ_h is the surface gravity, and $\mathcal{A}_h$ is the area of the black hole. When evaluating the boundary term on the black hole horizon we have assumed that it is a bifurcate Killing horizon and made use of the fact that the Killing generator vanishes on the bifurcation sphere.

Hence for $\Lambda \leq 0$, we see that taking $\hat{V}_{in}$ to be the black hole horizon and $\hat{V}_{out}$ the boundary at infinity, equation (2.9) with $\delta\Lambda \equiv 0$ gives the usual first law $\delta\mathcal{M} = \kappa_h\delta\mathcal{A}_h/(8\pi) + \Omega_h\delta\mathcal{J}$. We now turn to evaluating the boundary terms in (2.9) for different choices of $\hat{V}_{in}$ and $\hat{V}_{out}$ and now with the variation in the cosmological constant assumed to be non-vanishing.

B. First law in de Sitter with $\delta\Lambda$ and Smarr formula

There are several features of dS spacetime that are different from AdS or Minkowski that make the nature of a first law in dS distinct. Infinity in an asymptotically dS spacetime is a spacelike surface. Cosmological space-like slices asymptote to one corner of the conformal diagram. In Schwarzchild-like coordinates, surfaces of constant t_{schw} become timelike outside the cosmological horizon, and end at spacelike infinity. Likewise, the static Killing field $\xi^a = (\partial/\partial t_{schw})$ is space-like outside the horizon to infinity. One can use this Killing field to define an ADM charge at infinity, but it does not have the usual interpretation of a mass in the sense of Noether's theorem. On the other hand ξ^a is timelike between the black hole and cosmological horizon. This situation leads to three natural first law constructions in a black hole dS spacetime—from the black hole horizon to infinity, from the cosmological horizon to infinity, and between the two horizons. Only two of these are independent, as the construction between the horizons is the difference between the other two.

We first consider the first law construction outlined above for the region $\hat{V}$ contained between the black hole and cosmological horizons, rather than running the integrals out to infinity as in the asymptotically flat or AdS cases. This will give a relation between the variations of the two horizon areas, and does not include a mass parameter.

Let us first consider the contribution generated by the Killing vector $(\partial/\partial\phi)^a$. The variation δJ in the angular momentum is still given by equation (2.11), but now evaluated in an asymptotically de Sitter spacetime. Using equation (2.9) with the inner boundary taken to be the de Sitter horizon and the outer boundary at infinity one has

$$16\pi\delta\mathcal{J} = \int_\infty dar_c B^c[\partial/\partial\phi] = \int_{dS} dar_c B^c[\partial/\partial\phi] . \qquad (2.13)$$

So δJ is also given by evaluating the boundary integral on the de Sitter horizon. The Killing potential term does not contribute because we can connect the boundaries with a slice having normal n_a proportional to $\nabla_a t$, and hence the lapse function $F = -n_a(\partial/\partial\phi)^a$ in equation (2.6) vanishes.

Substituting the generator of the black hole horizon $(\partial/\partial t)^a + \Omega_h(\partial/\partial\phi)^a$ into the boundary integrand on the black hole gives the same result as in the asymptotically flat case, so that equation (2.12) continues to hold. The generator of the de Sitter horizon $(\partial/\partial t)^a + \Omega_c(\partial/\partial\phi)^a$ substituted into the boundary term on the de Sitter horizon gives a similar result

$$2|\kappa_c|\delta\mathcal{A}_c = \int_{dS} dar_a B^a[\partial/\partial t + \Omega_c\partial/\partial\phi] , \qquad (2.14)$$

with appropriate care taken for the signs. We have introduced the explicit absolute value signs on κ_c for clarity, since the definition of surface gravity gives $\kappa_c < 0$. Finally, noting that the boundary vector (2.5) is linear in the Killing field, one finds that the integral of the boundary term for the black hole horizon generator over the de Sitter horizon may be written as

$$\int_{dS} dar_a B^a[\partial/\partial t + \Omega_h\partial/\partial\phi] = \qquad (2.15)$$
$$2|\kappa_c|\delta\mathcal{A}_c + 16\pi(\Omega_h - \Omega_c)\delta\mathcal{J} .$$

The derivation of the first law is then completed by substituting the generator of the black hole horizon $\partial/\partial t + \Omega_h \partial/\partial\phi$ into the Gauss' law integral identity (2.9) and using (2.12) and (2.15) to arrive at

$$\frac{\kappa_h \delta \mathcal{A}_h}{8\pi} + \frac{|\kappa_c| \delta \mathcal{A}_c}{8\pi} + \frac{V \delta \Lambda}{8\pi} + (\Omega_h - \Omega_c)\delta\mathcal{J} = 0, \quad (2.16)$$

where the positive [cf. expression (3.15) below] thermodynamic volume V is defined by

$$V = -\left(\int_{dS} dar_c n_b \omega^{cb} - \int_{bh} dar_c n_b \omega^{cb} \right). \quad (2.17)$$

The Smarr relation for rotating black holes in de Sitter spacetime follows by integrating the first law under a scaling transformation. The scaling dimensions of $\mathcal{A}$ and $\mathcal{J}$ are $D-2$, while Λ has dimension -2, giving

$$\frac{\kappa_h \mathcal{A}_h}{8\pi} + \frac{|\kappa_c| \mathcal{A}_c}{8\pi} + (\Omega_h - \Omega_c)\mathcal{J} = \frac{V \Lambda}{4\pi(D-2)}. \quad (2.18)$$

Let us now identify the positive cosmological constant Λ with the negative thermodynamic pressure as in (1.1), $P = -\frac{\Lambda}{8\pi} < 0$, commensurate with the definition in the anti de Sitter case [13]. With this definition we get the following forms for the first law and the Smarr relation respectively (including the possibility of more than one rotation parameter)

$$0 = T_h \delta S_h + T_c \delta S_c$$
$$+ \sum_i (\Omega_h^i - \Omega_c^i)\delta J^i - V \delta P, \quad (2.19)$$

$$0 = T_h S_h + T_c S_c$$
$$+ \sum_i (\Omega_h^i - \Omega_c^i)J^i + \frac{2}{D-2}PV, \quad (2.20)$$

where we identified the horizon area with the entropy $S = \frac{\mathcal{A}}{4}$ and have defined the cosmological and black hole temperatures to be the positive quantities $T_c = \frac{|\kappa_c|}{2\pi}$ and $T_h = \frac{\kappa_h}{2\pi}$.

C. Going to infinity

One can essentially repeat the derivation of section II B in two additional cases: i) one takes the slice to go from the black hole horizon to infinity (passing through the dS horizon), and ii) the slice extends from the cosmological horizon to infinity. Consequently one obtains two additional first law relations and associated Smarr formulae, one for each horizon. In both cases, the expressions involve a quantity M which would be the ADM mass in the flat and AdS cases. In the dS case however, such a quantity is "conserved in space" (rather than in time) due to the spacelike character of the Killing field $(\partial/\partial t)^a$ in the region near infinity. Keeping this (important) distinction in mind, we shall refer to M as the "mass" [20], and use it as a book-keeping device.

For the black hole horizon we then get the first law relation

$$\delta M = T_h \delta S_h + \sum_i (\Omega_h^i - \Omega_\infty^i)\delta J^i + V_h \delta P, \quad (2.21)$$

where the quantities Ω_∞^i allow for the possibility of a rotating frame at infinity (see, e.g., reference [34]), δM is given via the boundary integral

$$16\pi\,\delta M = -\int_\infty dar_c \left(B^c[\partial/\partial t] + 2\,\delta\Lambda\,\omega_{dS}^{cd} n_d \right), \quad (2.22)$$

and the thermodynamic volume V_h is given by the expression

$$V_h = \int_\infty dar_c n_d(\omega^{cd} - \omega_{dS}^{cd}) - \int_{bh} dar_c n_d \omega^{cd}. \quad (2.23)$$

The quantity ω_{dS}^{ab} appearing in equations (2.22) and (2.23) is the Killing potential of the background de Sitter spacetime, without the black hole. These contributions serve to make each of the quantities δM and V_h finite. See reference [3] for an in depth discussion of this point in the asymptotically AdS case. The corresponding Smarr formula, obtained from the first law by overall scaling, is then given by

$$\frac{D-3}{D-2}M = T_h S_h + \sum_i (\Omega_h^i - \Omega_\infty^i)J^i - \frac{2}{D-2}PV_h, \quad (2.24)$$

We see from the first law (2.21) that the thermodynamic volume V_h may be interpreted as the change in the mass under variations in the cosmological constant with the black hole horizon area and angular momentum held fixed.

For the cosmological horizon, we use the Gauss' law relation (2.9) with the corresponding horizon generating Killing vector $(\partial/\partial t)^a + \Omega_c(\partial/\partial\phi)^a$ to obtain the first law

$$\delta M = -T_c \delta S_c + \sum_i (\Omega_c^i - \Omega_\infty^i)\delta J^i + V_c \delta P. \quad (2.25)$$

The minus sign in the first term on the right hand side arises because the surface gravity κ_c of the de Sitter horizon is negative, while the corresponding temperature T_c is taken to be *positive*. The thermodynamic volume V_c for the cosmological horizon is given by

$$V_c = \int_\infty dar_e n_f(\omega^{ef} - \omega_{dS}^{ef}) - \int_{dS} dar_e n_f \omega^{ef}. \quad (2.26)$$

Finally, the Smarr relation that follows from (2.25) by scaling is given by

$$\frac{D-3}{D-2}M = -T_c S_c + \sum_i (\Omega_c^i - \Omega_\infty^i)J^i - \frac{2}{D-2}PV_c, \quad (2.27)$$

We see from the first law (2.25) that the thermodynamic volume V_c gives the variation in the mass assuming that

the area of the cosmological horizon, as well as the angular momentum, is held fixed, which is distinct from the conditions defining the thermodynamic volume V_h.

The first law (2.19) discussed in the previous section, which concerns only the region between the black hole and cosmological horizons, can now be understood as a consequence of subtracting the formulas (2.21) and (2.25). We see from the explicit expressions of the thermodynamic volumes V in (2.17) and V_h, V_c in (2.23) and (2.26) that they satisfy the relation

$$V = V_c - V_h \,. \tag{2.28}$$

The Smarr formula (2.20) for the region between the horizons is similarly given by the difference between equations (2.24) and (2.27).

III. THERMODYNAMICS OF KERR-DESITTER BLACK HOLES.

We now turn to the example of Kerr-deSitter black holes. The D-dimensional Kerr-(A)dS spacetimes [35, 36] are solutions to the Einstein equations

$$R_{ab} = \frac{2\Lambda}{(D-2)} g_{ab}$$

that generalize the asymptotically-flat rotating black hole solutions of [37]. The thermodynamics for the Kerr-AdS case was studied in [34]. In the asymptotically dS case the metric in 'generalized' Boyer-Lindquist coordinates takes the form

$$ds^2 = -W(1 - g^2 r^2) dt^2 + \frac{2m}{U}\Big(W dt - \sum_{i=1}^{N} \frac{a_i \mu_i^2 d\phi_i}{\Xi_i}\Big)^2$$

$$+ \sum_{i=1}^{N} \frac{r^2 + a_i^2}{\Xi_i} (\mu_i^2 d\phi_i^2 + d\mu_i^2) + \frac{U dr^2}{X - 2m} + \epsilon r^2 d\nu^2$$

$$+ \frac{g^2}{W(1 - g^2 r^2)} \Big(\sum_{i=1}^{N} \frac{r^2 + a_i^2}{\Xi_i} \mu_i d\mu_i + \epsilon r^2 \nu d\nu\Big)^2, \tag{3.1}$$

where $2\Lambda = (D-1)(D-2)g^2$ and

$$W = \sum_{i=1}^{N} \frac{\mu_i^2}{\Xi_i} + \epsilon \nu^2, \quad X = r^{\epsilon-2}(1 - g^2 r^2) \prod_{i=1}^{N}(r^2 + a_i^2),$$

$$U = \frac{Z}{1 - g^2 r^2}\Big(1 - \sum_{i=1}^{N} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}\Big), \quad \Xi_i = 1 + g^2 a_i^2. \tag{3.2}$$

Here $N \equiv [(D-1)/2]$, where $[A]$ means the integer part of A, and we have defined ϵ to be 1 for D even and 0 for odd. The coordinates μ_i are not independent, but obey the constraint

$$\sum_{i=1}^{N} \mu_i^2 + \epsilon \nu^2 = 1\,. \tag{3.3}$$

A. Even dimensions

In even dimensions ($D = 2N + 2$), the thermodynamic quantities are calculated as follows. For the cosmological horizon, we have

$$S_c = \tfrac{1}{4}\mathcal{A}_{D-2} \prod_i \frac{r_c^2 + a_i^2}{\Xi_i}\,,$$

$$T_c = -\frac{r_c(1 - g^2 r_c^2)}{2\pi} \sum_i \frac{1}{r_c^2 + a_i^2} + \frac{1 + g^2 r_c^2}{4\pi r_c}\,,$$

$$\Omega_c^i = \frac{(1 - g^2 r_c^2) a_i}{r_c^2 + a_i^2}\,, \tag{3.4}$$

while the 'mass' and angular momenta read

$$M = \frac{m \mathcal{A}_{D-2}}{4\pi \prod_j \Xi_j} \sum_i \frac{1}{\Xi_i}\,, \quad J_i = \frac{m a_i \mathcal{A}_{D-2}}{4\pi \Xi_i \prod_j \Xi_j}\,, \tag{3.5}$$

where the cosmological horizon radius r_c and black hole horizon radius r_h are solutions to

$$2m = \frac{1}{r_c}(1 - g^2 r_c^2) \prod_i (r_c^2 + a_i^2) = \frac{1}{r_h}(1 - g^2 r_h^2) \prod_i (r_h^2 + a_i^2)\,. \tag{3.6}$$

The quantity $\mathcal{A}_{D-2}$ is the volume of the unit-radius $(D-2)$-sphere, and is given by

$$\mathcal{A}_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}\,. \tag{3.7}$$

Using the Smarr relation (2.27) one finds that the thermodynamic volume V_c associated with the de Sitter horizon is given by

$$V_c = \frac{r_c A_c}{D-1}\left[1 + \frac{1 - g^2 r_c^2}{(D-2)r_c^2} \sum_i \frac{a_i^2}{\Xi_i}\right] \tag{3.8}$$

$$= \frac{r_c A_c}{D-1} + \frac{8\pi}{(D-1)(D-2)} \sum_i a_i J_i\,. \tag{3.9}$$

Given that the Smarr formula (2.27) was derived from the first law (2.25), it also follows that the quantities (3.4), (3.5), and (3.8) satisfy the cosmological horizon first law.

Similarly, for the black hole horizon we have

$$S_h = \tfrac{1}{4}\mathcal{A}_{D-2} \prod_i \frac{r_h^2 + a_i^2}{\Xi_i}\,,$$

$$T_h = \frac{r_h(1 - g^2 r_h^2)}{2\pi} \sum_i \frac{1}{r_h^2 + a_i^2} - \frac{1 + g^2 r_h^2}{4\pi r_h}\,,$$

$$\Omega_h^i = \frac{(1 - g^2 r_h^2) a_i}{r_h^2 + a_i^2}\,, \tag{3.10}$$

while the thermodynamic volume calculated from (2.24) reads

$$V_h = \frac{r_h A_h}{D-1}\left[1 + \frac{1 - g^2 r_h^2}{(D-2)r_h^2} \sum_i \frac{a_i^2}{\Xi_i}\right] \tag{3.11}$$

$$= \frac{r_h A_h}{D-1} + \frac{8\pi}{(D-1)(D-2)} \sum_i a_i J_i\,. \tag{3.12}$$

The horizon quantities (3.10), (3.5), and (3.11) then satisfy the black hole horizon first law (2.21).

The expressions for the thermodynamic volumes V_c in (3.9) and V_h in (3.12) have the same form as one another and also as the result for V_h in the Kerr-AdS case [8]. As in [8] the first terms in these expressions for V_c and V_h are equal to geometric volumes, respectively denoted by

$$V_c' = \frac{r_c A_c}{D-1}, \qquad V_h' = \frac{r_h A_h}{D-1}, \qquad (3.13)$$

contained within the horizon, which are obtained by integrating the full D-dimensional volume element over the region on a t equals constant slice between $r = 0$ and the horizon radius.[3]

We see from (3.9) and (3.12) that the thermodynamic volumes V_c and V_h differ from their geometric counterparts V_c' and V_h' by precisely the same amount. It then follows that the thermodynamic volume $V = V_c - V_h$ that enters the first law (2.19) and Smarr relation (2.20) for the region between the two horizons is exactly equal to its geometric counterpart $V' = V_c' - V_h'$, with

$$V = V' = \frac{r_c A_c}{D-1} - \frac{r_h A_h}{D-1}. \qquad (3.14)$$

In particular, for the Schwarzschild-dS spacetimes we get the following 'manifest' geometric relation

$$V = \frac{A_{D-2}}{D-1} \left(r_c^{D-1} - r_h^{D-1} \right). \qquad (3.15)$$

The quantities (3.4), (3.5), (3.10), and (3.14) satisfy the first law of black hole thermodynamics (2.19) and the Smarr relation (2.20).

B.   Odd dimensions

In odd dimensions $(D = 2N + 1)$, the following thermodynamic quantities get modified: the mass

$$M = \frac{m A_{D-2}}{4\pi \prod_j \Xi_j} \left(\sum_i \frac{1}{\Xi_i} - \frac{1}{2} \right), \qquad (3.16)$$

the cosmological horizon entropy and temperature

$$S_c = \frac{A_{D-2}}{4r_c} \prod_i \frac{r_c^2 + a_i^2}{\Xi_i},$$

$$T_c = -\frac{r_c(1 - g^2 r_c^2)}{2\pi} \sum_i \frac{1}{r_c^2 + a_i^2} + \frac{1}{2\pi r_c},$$

----

[3] A similar formula holds also in odd dimensions, but the integration now proceeds between $r^2 = -a_{min}^2$ where a_{min}^2 is the smallest among the values of the squares of the rotational parameters a_i^2. Note that in an even dimension such a_{min}^2 automatically equals zero.

and the black hole horizon entropy and temperature

$$S_h = \frac{A_{D-2}}{4r_h} \prod_i \frac{r_h^2 + a_i^2}{\Xi_i},$$

$$T_h = \frac{r_h(1 - g^2 r_h^2)}{2\pi} \sum_i \frac{1}{r_h^2 + a_i^2} - \frac{1}{2\pi r_h}. \qquad (3.17)$$

The other quantities, including Ω_h^i and Ω_c^i, and J_i remain of the same form, with m related to the cosmological and black hole horizon radii according to

$$2m = \frac{1}{r_c^2} (1 - g^2 r_c^2) \prod_i (r_c^2 + a_i^2) = \frac{1}{r_h^2} (1 - g^2 r_h^2) \prod_i (r_h^2 + a_i^2). \qquad (3.18)$$

It is easy to verify that the thermodynamic volumes V, V_c and V_h again take the form (3.14), (3.8) and (3.11) and that all the quantities satisfy the Smarr relations (2.20), (2.24), (2.27) and the first laws (2.19), (2.21) and (2.25).

IV.   ISOPERIMETRIC INEQUALITIES

A.   Euclidean space

The isoperimetric inequality for the volume $\mathcal{V}$ of a connected domain in Euclidean space $\mathbb{E}^{D-1}$ whose area is $\mathcal{A}$ states that the ratio

$$R = \left(\frac{(D-1)\mathcal{V}}{A_{D-2}} \right)^{\frac{1}{D-1}} \left(\frac{A_{D-2}}{\mathcal{A}} \right)^{\frac{1}{D-2}} \qquad (4.1)$$

obeys $R \le 1$, with equality if and only if the domain is a standard round ball. That is, for a fixed volume, the area that surrounds the volume is minimized when the volume is a ball.

B.   Reverse isoperimetric inequalities for V_h and V_c

Kerr-AdS black holes have been shown to satisfy a 'reverse' isoperimetric inequality [8], with the thermodynamic volume bounded from below, rather than from above, in relation to the horizon area. We find similar results in this subsection for the thermodynamic volumes V_h and V_c associated with the black hole and de Sitter horizons. Both satisfy reverse isoperimetric inequalities in relation to the corresponding horizon areas. Novel results will arise, however, when we consider the region between the horizons. We will find that a true isoperimetric inequality bounds the thermodynamic volume V of this region from above in terms of the horizon areas.

Consider first the black hole horizon where we take the volume $\mathcal{V}$ entering the isoperimetric inequality to be given by the thermodynamic volume of the black hole horizon

$$\mathcal{V} \equiv V_h = \frac{r_h A_h}{D-1} \left[1 + \frac{1 - g^2 r_h^2}{(D-2)r_h^2} \sum_i \frac{a_i^2}{\Xi_i} \right], \qquad (4.2)$$

and the area $\mathcal{A}$ to be the black hole horizon area

$$\mathcal{A} = A_h = \begin{cases} \frac{\mathcal{A}_{D-2}}{r_h} \prod_i \frac{r_h^2 + a_i^2}{\Xi_i}, & D \text{ odd} \\ \mathcal{A}_{D-2} \prod_i \frac{r_h^2 + a_i^2}{\Xi_i}, & D \text{ even}. \end{cases} \quad (4.3)$$

The statement now is that the corresponding ratio R defined in (4.1), similar to the AdS case, satisfies the *reverse isoperimetric inequality*,

$$R \geq 1, \quad (4.4)$$

with equality if and only if there is no rotation. This may be rephrased as stating that *for Kerr-dS spacetimes with a fixed thermodynamic volume V_h the black hole entropy $A_h/4$ is maximized for Schwarzschild-dS.* As in the AdS case [8], we conjecture that a similar statement holds for any asymptotically dS black hole.

The proof of (4.4) for Kerr-dS follows closely the one for Kerr-AdS in [8]. We define a dimensionless quantity

$$z = \frac{(1 - g^2 r_h^2)}{r_h^2} \sum_i \frac{a_i^2}{\Xi_i}, \quad (4.5)$$

and consider R^{D-1}. In odd dimensions we find

$$\begin{aligned} R^{D-1} &= r_h \left[1 + \frac{z}{D-2} \right] \left[\frac{1}{r_h} \prod_i \frac{(r_h^2 + a_i^2)}{\Xi_i} \right]^{-\frac{1}{D-2}} \\ &= \left[1 + \frac{z}{D-2} \right] \left[\prod_i \frac{(r_h^2 + a_i^2)}{r_h^2 \Xi_i} \right]^{-\frac{1}{D-2}} \\ &\geq \left[1 + \frac{z}{D-2} \right] \left[\frac{2}{D-1} \left(\sum_i \frac{1}{\Xi_i} + \sum_i \frac{a_i^2}{r_h^2 \Xi_i} \right) \right]^{-\frac{(D-1)}{2(D-2)}} \\ &= \left[1 + \frac{z}{D-2} \right] \left[1 + \frac{2z}{D-1} \right]^{-\frac{(D-1)}{2(D-2)}} \equiv F(z), \quad (4.6) \end{aligned}$$

where the inequality follows from the AG inequality (inequality relating the arithmetic and geometric means)

$$\left(\prod_i x_i \right)^{1/N} \leq \frac{1}{N} \sum_i x_i \quad (4.7)$$

for non-negative quantities x_i, and the equality follows from (4.5) and the definition $\Xi_i = 1 + g^2 a_i^2$. Noting that $F(0) = 1$, and that

$$\frac{d \log F(z)}{dz} = \frac{(D-3) z}{(D-2)(D-2+z)(D-1+2z)}, \quad (4.8)$$

which is positive for non-negative z in $D > 3$ dimensions, it follows that $F(z) \geq 1$, and hence the reverse isoperimetric inequality (4.4) is satisfied by all odd-dimensional Kerr-dS black holes.

In even dimensions the calculation is similar. One finds

that

$$\begin{aligned} R^{D-1} &= r_h \left[1 + \frac{z}{D-2} \right] \left[\prod_i \frac{(r_h^2 + a_i^2)}{\Xi_i} \right]^{-\frac{1}{D-2}} \\ &= \left[1 + \frac{z}{D-2} \right] \left[\prod_i \frac{(r_h^2 + a_i^2)}{r_h^2 \Xi_i} \right]^{-\frac{1}{D-2}} \\ &\geq \left[1 + \frac{z}{D-2} \right] \left[\frac{2}{D-2} \left(\sum_i \frac{1}{\Xi_i} + \sum_i \frac{a_i^2}{r_h^2 \Xi_i} \right) \right]^{-\frac{1}{2}} \\ &= \left[1 + \frac{z}{D-2} \right] \left[1 + \frac{2z}{D-2} \right]^{-\frac{1}{2}} \equiv G(z). \quad (4.9) \end{aligned}$$

Thus $G(0) = 1$ and $d \log G(z)/dz \geq 0$, and so again we conclude that $R \geq 1$. Thus the reverse isoperimetric inequality holds for even-dimensional Kerr-dS black holes also. It further follows via direct substitution that identical results hold for the cosmological thermodynamic volume V_c and horizon area A_c. One can therefore state that *for Kerr-dS spacetimes having a fixed value of the cosmological thermodynamic volume V_c, the cosmological horizon entropy $A_c/4$ is maximized by the Schwarzschild-dS spacetime.*

C. True isoperimetric inequality for the thermodynamic volume V between the horizons

Let us now focus on the thermodynamic volume V of the region between the black hole and de Sitter horizon. Recall that in section III we found that for Kerr-deSitter spacetimes V coincides with the geometric volume V' between the horizons. We now want to ask whether V satisfies some sort of inequality with respect to the horizon areas.

Consider the Kerr-dS black hole in any dimension. For the purposes of establishing a bound, we will choose to work with the volume parameter $\mathcal{V}$ given by the geometric volume of the de Sitter horizon,

$$\mathcal{V} \equiv \frac{r_c A_c}{D-1} \geq V = \frac{r_c A_c}{D-1} - \frac{r_h A_h}{D-1}, \quad (4.10)$$

which is manifestly greater than or equal to the thermodynamic volume V. For the area $\mathcal{A}$ we take the area of the de Sitter horizon

$$\mathcal{A} \equiv A_c \leq A_c + A_h \equiv A \quad (4.11)$$

which is less than or equal to the total area A of the black hole and de Sitter horizons. With these choices for $\mathcal{V}$ and $\mathcal{A}$ we then find, following [8], that the ratio R defined by (4.1) is given in all dimensions D by

$$R = \left(\prod_i R_i \right)^{-\frac{1}{(D-1)(D-2)}}, \quad R_i = \frac{1 + a_i^2/r_c^2}{1 + g^2 a_i^2}. \quad (4.12)$$

Since one always has $\frac{1}{r_c^2} \geq g^2$ each term $R_i \geq 1$; consequently

$$R \leq 1. \quad (4.13)$$

On the one hand, this inequality might look trivial, since the volume and area being compared are as in Euclidean space. The interesting thing is that this simple geometric volume is what arises in the Smarr relation, and that this is true even with rotation.

The quantities $\mathcal{V}$ and $\mathcal{A}$, and hence also the quantities V and A, therefore satisfy a *true isoperimetric inequality. Equality , $R = 1$, is attained in* (4.13) if and only there is no black hole ($\mathcal{V} = V$), in which case $r_c^2 = 1/g^2$. Since in this case we also have $\mathcal{A} = A$ we may formulate the following result: *For a fixed thermodynamic volume V in between the black hole and dS horizons the total entropy $S = A/4$ is minimized if there is no black hole.* Note that if instead choosing $(\mathcal{V}, \mathcal{A})$ as in (4.10) and (4.11) we chose $\mathcal{V} = V$ and $\mathcal{A} = A$ the departure of R from equality would be even more severe.

We have seen that the thermodynamic volume V between the two horizons is the same as the geometrical volume V', even with rotation. So fixing V is like fixing the size of the observable universe. Hence another way of stating the isoperimetric inequality is that for a given size of the observable universe, the entropy interior to the cosmological horizon is increased by adding a black hole, even though the black hole pulls in the cosmological horizon [40]. By assuming that non-black hole inhomogeneities follow the same entropy rule, this entropy increase has been used to estimate the probability that inflation occurs for a universe in "the landscape" [39]. It is interesting that V' arises naturally as part of the free energy balance between the cosmological and black hole horizons, and that it applies to black holes with angular momentum.

Lastly, we emphasize that this situation is opposite to asymptotically AdS or flat case [8], where the *reverse isoperimetric inequality* for the thermodynamic volume was proved for Kerr-AdS black holes in any dimension. We conjecture that this feature may remain valid for any asymptotically de Sitter black hole.

V. CHARGED BLACK HOLES

In this section, we will extend our analysis to various charged and rotating de Sitter black hole solutions. It is straightforward to extend the Hamiltonian perturbation theory analysis of section II to include $U(1)$ charges in the first laws and Smarr relations holding for different sets of boundaries. We then consider the examples of the Kerr-Newman-deSitter solution in $D = 4$ and the de Sitter versions of the Chong-Cvetic-Lu-Pope charged, rotating solutions in minimal $D = 5$ supergravity [28], which includes a Chern-Simons interaction for the $U(1)$ gauge field.

For Kerr-Newman black holes, we find results that match those of section III. The black hole and cosmological thermodynamic volumes V_h and V_c each satisfy reverse isoperimetric inequalities. The thermodynamic volume V is again equal to the geometric volume V' for

the region between the horizons and again satisfies a true isoperimetric inequality. For Chong-Cvetic-Lu-Pope-dS black holes our results are less complete. Whereas we are able to show analytically that a similar set of inequalities hold subject to a certain limitation on the range of the charge and rotational parameters, in the general case we have only numerical support for this claim. Equality of the thermodynamic volume V with the geometric volume V' between the horizons, however, is shown to hold over the entire parameter range.

We find that the modifications to the first laws and Smarr relations for the various sets of boundaries are as one would expect. For the region between the black hole and de Sitter horizon the new formulas are given by

$$0 = T_h \delta S_h + T_c \delta S_c + \sum_j (\Phi_h^j - \Phi_c^j)\delta Q^j$$
$$+ \sum_i (\Omega_h^i - \Omega_c^i)\delta J^i - V\delta P, \quad (5.1)$$

$$0 = T_h S_h + T_c S_c + \frac{D-3}{D-2}\sum_j (\Phi_h^j - \Phi_c^j)Q^j$$
$$+ \sum_i (\Omega_h^i - \Omega_c^i)J^i + \frac{2}{D-2}VP, \quad (5.2)$$

where Φ_h^i and Φ_c^i are the potentials for the electric (and magnetic) $U(1)$ charges evaluated at the black hole and de Sitter horizons.

For the regions stretching respectively from the black hole and de Sitter horizons out to infinity we have the first laws and Smarr relations

$$\delta M = T_h \delta S_h + \sum_j (\Phi_h^j - \Phi_\infty^j)\delta Q^j$$
$$+ \sum_i (\Omega_h^i - \Omega_\infty^i)\delta J^i + V_h\delta P, \quad (5.3)$$

$$\frac{D-3}{D-2}M = T_h S_h + \frac{D-3}{D-2}\sum_j (\Phi_h^j - \Phi_\infty^j)Q^j$$
$$+ \sum_i (\Omega_h^i - \Omega_\infty^i)J^i - \frac{2}{D-2}V_h P, \quad (5.4)$$

and

$$\delta M = -T_c \delta S_c + \sum_j (\Phi_c^j - \Phi_\infty^j)\delta Q^j$$
$$+ \sum_i (\Omega_c^i - \Omega_\infty^i)\delta J^i + V_c\delta P, \quad (5.5)$$

$$\frac{D-3}{D-2}M = -T_c S_c + \frac{D-3}{D-2}\sum_j (\Phi_c^j - \Phi_\infty^j)Q^j$$
$$+ \sum_i (\Omega_c^i - \Omega_\infty^i)J^i - \frac{2}{D-2}V_c P. \quad (5.6)$$

where additionally the quantities Φ_∞^i are the values of the electric and magnetic potentials at infinity. As before, subtracting respectively (5.6) from (5.4) yields (5.2), and subtracting respectively (5.5) from (5.3) yields (5.1).

A. Kerr-Newman-dS black hole

The $D = 4$ Kerr-Newman-deSitter metric for a rotating, charged black hole with positive cosmological constant reads

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2$$
$$+\frac{\rho^2}{S}d\theta^2 + \frac{S\sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\varphi\right)^2, \quad (5.7)$$

where the various functions entering the metric are given by

$$\Delta = (r^2 + a^2)(1 - r^2 g^2) - 2mr + \hat{z}^2 \quad (5.8)$$
$$S = 1 + a^2 g^2 \cos^2\theta, \quad \hat{z}^2 = q_e^2 + q_m^2, \quad (5.9)$$
$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Xi = 1 + a^2 g^2, \quad (5.10)$$

and the vector potential is

$$A = -\frac{q_e r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right) - \frac{q_m\cos\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\varphi\right).$$
$$(5.11)$$

The thermodynamic quantities in Kerr-Newman-AdS spacetimes were computed in [2]. From those results we can infer the following formulas in the de Sitter case. For the cosmological horizon we find

$$S_c = \frac{\pi(r_c^2 + a^2)}{\Xi}, \quad \Omega_c = \frac{a\Xi}{r_c^2 + a^2} \quad (5.12)$$

$$T_c = -\frac{r_c\left(1 - a^2 g^2 - 3g^2 r_c^2 - \frac{a^2+\hat{z}^2}{r_c^2}\right)}{4\pi(r_c^2 + a^2)}, \quad (5.13)$$

$$\Phi_c^{(e)} = \frac{q_e r_c}{r_c^2 + a^2}, \quad \Phi_c^{(m)} = \frac{q_m r_c}{r_c^2 + a^2}, \quad (5.14)$$

where for magnetic charge we take the magnetic potential analogous to the electric one [38]. The mass, the electric and magnetic charges are given by

$$M = \frac{m}{\Xi^2}, \quad Q_e = \frac{q_e}{\Xi}, \quad Q_m = \frac{q_m}{\Xi}, \quad (5.15)$$

while the angular momentum and the angular velocity at infinity read

$$J = \frac{am}{\Xi^2}, \quad \Omega_\infty = ag^2. \quad (5.16)$$

From the Smarr formula (5.6) we compute the cosmological thermodynamic volume

$$V_c = \frac{r_c A_c}{3}\left[1 + \frac{a^2}{2\Xi r_c^2}\left(1 - g^2 r_c^2 + \frac{\hat{z}^2}{r_c^2 + a^2}\right)\right], \quad (5.17)$$

and easily verify that the first law (5.5) is satisfied. Since V_c in the charged case is greater than or equal to the uncharged one, while the A_c has the same form, the reverse isoperimetric inequality for the cosmological thermodynamic volume (4.4) remains valid.

Similarly, for the black hole horizon we have the following expressions for the thermodynamic quantities

$$S_h = \frac{\pi(r_h^2 + a^2)}{\Xi}, \quad \Omega_h = \frac{a\Xi}{r_h^2 + a^2} \quad (5.18)$$

$$T_h = \frac{r_h\left(1 - a^2 g^2 - 3g^2 r_h^2 - \frac{a^2+\hat{z}^2}{r_h^2}\right)}{4\pi(r_h^2 + a^2)}, \quad (5.19)$$

$$\Phi_h^{(e)} = \frac{q_e r_h}{r_h^2 + a^2}, \quad \Phi_h^{(m)} = \frac{q_m r_h}{r_h^2 + a^2}. \quad (5.20)$$

The black hole thermodynamic volume is then given by the expression

$$V_h = \frac{r_h A_h}{3}\left[1 + \frac{a^2}{2\Xi r_h^2}\left(1 - g^2 r_h^2 + \frac{\hat{z}^2}{r_h^2 + a^2}\right)\right] \quad (5.21)$$

which also satisfies the reverse isoperimetric inequality (4.4).

Finally, the thermodynamic volume V in between the horizons calculated from (5.2) again takes a simple geometric form,

$$V = \frac{1}{3}\left(r_c \mathcal{A}_c - r_h \mathcal{A}_h\right). \quad (5.22)$$

Defining the volume $\mathcal{V}$ and area $\mathcal{A}$ as in Section IV C to be

$$\mathcal{V} \equiv \frac{r_c A_c}{3} \geq V, \quad \mathcal{A} \equiv A_c \leq A_c + A_h \equiv A, \quad (5.23)$$

we find that the isoperimetric inequality (4.13) holds also for Kerr–Newman-deSitter black holes.

B. Charged-dS rotating black hole in $D = 5$

The rotating charged-AdS black hole with electromagnetic Chern-Simons term was constructed by Chong, Cvetic, Lu, and Pope [28]. We find that a corresponding rotating charged-dS black hole exists as an exact solution to the Einstein-Chern-Simons equations with positive cosmological constant, obtained via analytic continuation $g \to ig$. This black hole solution reads

$$ds^2 = -\frac{S\left[(1 + g^2 r^2)\rho^2 dt + 2q\nu\right]dt}{\Xi_a \Xi_b \rho^2} + \frac{2q\nu\omega}{\rho^2}$$
$$+\frac{f}{\rho^4}\left(\frac{Sdt}{\Xi_a \Xi_b} - \omega\right)^2 + \frac{\rho^2 dr^2}{\Delta} + \frac{\rho^2 d\theta^2}{S}$$
$$+\frac{r^2 + a^2}{\Xi_a}\sin^2\theta d\phi^2 + \frac{r^2 + b^2}{\Xi_b}\cos^2\theta d\psi^2,$$

$$A = \frac{\sqrt{3}q}{\rho^2}\left(\frac{Sdt}{\Xi_a \Xi_b} - \omega\right), \quad (5.24)$$

where

$$\nu = b\sin^2\theta d\phi + a\cos^2\theta d\psi\,,$$
$$\omega = a\sin^2\theta\frac{d\phi}{\Xi_a} + b\cos^2\theta\frac{d\psi}{\Xi_b}\,,$$
$$S = 1 + a^2g^2\cos^2\theta + b^2g^2\sin^2\theta\,,$$
$$\Delta = \frac{(r^2+a^2)(r^2+b^2)(1-g^2r^2)+q^2+2abq}{r^2} - 2m\,,$$
$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,,$$
$$\Xi_a = 1 + a^2g^2\,,\quad \Xi_b = 1 + b^2g^2\,,$$
$$f = 2m\rho^2 - q^2 - 2abqg^2\rho^2\,. \tag{5.25}$$

In the case of a negative cosmological constant [28], the solution describes an asymptotically AdS charged rotating black hole in a certain range of parameters m, a, b and q. It is easy to check that there exists a range for these parameters in which the function Δ in Eq. (5.25), admits three positive real roots. Hence in this range the solution describes an asymptotically dS charged rotating back hole. To our knowledge this is the first demonstration of the existence of such black hole solutions.

The thermodynamic quantities are [28]

$$M = \frac{\pi m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) - 2\pi qabg^2(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2}\,,$$
$$J^a = \frac{\pi(2am + qb(1-a^2g^2)]}{4\Xi_a^2\Xi_b}\,,$$
$$J^b = \frac{\pi(2bm + qa(1-b^2g^2)]}{4\Xi_b^2\Xi_a}\,,$$
$$Q = \frac{\sqrt3\,\pi q}{4\Xi_a\Xi_b}\,. \tag{5.26}$$

For the black hole horizon we have

$$T_h = \frac{r_h^4[1-g^2(2r_h^2+a^2+b^2)]-(ab+q)^2}{2\pi r_h[(r_h^2+a^2)(r_h^2+b^2)+abq]}\,,$$
$$S_h = \frac{\pi^2[(r_h^2+a^2)(r_h^2+b^2)+abq]}{2\Xi_a\Xi_b\,r_h}\,,$$
$$\Phi_h = \frac{\sqrt3\,qr_h^2}{(r_h^2+a^2)(r_h^2+b^2)+abq}\,,$$
$$\Omega_h^a = \frac{a(r_h^2+b^2)(1-g^2r_h^2)+bq}{(r_h^2+a^2)(r_h^2+b^2)+abq}\,,$$
$$\Omega_h^b = \frac{b(r_h^2+a^2)(1-g^2r_h^2)+aq}{(r_h^2+a^2)(r_h^2+b^2)+abq}\,. \tag{5.27}$$

The Smarr formula (5.4) then leads to an expression for the black hole thermodynamic volume

$$V_h = \frac{r_hA_h}{4}\left[1 + \frac{1-g^2r_h^2}{3r_h^2}\left(\frac{a^2}{\Xi_a}+\frac{b^2}{\Xi_b}\right)\right.$$
$$\left.+\frac{q^2d^2+abq(d^2+r_h^2-r_h^2a^2b^2g^4)}{3\Xi_a\Xi_br_h^2[(r_h^2+a^2)(r_h^2+b^2)+abq]}\right]. \tag{5.28}$$

where $d^2 = a^2 + b^2 + 2a^2b^2g^2$. When we switch off one of the rotations (*e.g.* by setting $a = 0$) the reverse isoperimetric inequality (4.4) still holds. However, because of

the complexity of the expressions we have not been able to analytically establish or disprove this result in the general case and we leave the question for future study. We have, nevertheless, confirmed the plausibility of this conjecture by a numerical study.

Similarly, for the cosmological horizon we have

$$T_c = -\frac{r_c^4[1-g^2(2r_c^2+a^2+b^2)]-(ab+q)^2}{2\pi r_c[(r_c^2+a^2)(r_c^2+b^2)+abq]}\,,$$
$$S_c = \frac{\pi^2[(r_c^2+a^2)(r_c^2+b^2)+abq]}{2\Xi_a\Xi_b\,r_c}\,,$$
$$\Phi_c = \frac{\sqrt3\,qr_c^2}{(r_c^2+a^2)(r_c^2+b^2)+abq}\,,$$
$$\Omega_c^a = \frac{a(r_c^2+b^2)(1-g^2r_c^2)+bq}{(r_c^2+a^2)(r_c^2+b^2)+abq}\,,$$
$$\Omega_c^b = \frac{b(r_c^2+a^2)(1-g^2r_c^2)+aq}{(r_c^2+a^2)(r_c^2+b^2)+abq}\,, \tag{5.29}$$

which gives

$$V_c = \frac{r_cA_c}{4}\left[1 + \frac{1-g^2r_c^2}{3r_c^2}\left(\frac{a^2}{\Xi_a}+\frac{b^2}{\Xi_b}\right)\right.$$
$$\left.+\frac{q^2d^2+abq(d^2+r_c^2-r_c^2a^2b^2g^4)}{3\Xi_a\Xi_br_c^2[(r_h^2+a^2)(r_c^2+b^2)+abq]}\right]. \tag{5.30}$$

Finally, using the Smarr relation (5.2), we find that the thermodynamic volume in between the horizons is

$$V = \frac{1}{4}(r_cA_c - r_hA_h)$$
$$= \frac{\pi^2}{2\Xi_a\Xi_b}\left[(r_c^2+a^2)(r_c^2+b^2)-(r_h^2+a^2)(r_h^2+b^2)\right], \tag{5.31}$$

which again, coincides with the naive geometric one. For $abq \geq 0$ and taking

$$\mathcal{V} = \frac{\pi^2}{2\Xi_a\Xi_b}(r_c^2+a^2)(r_c^2+b^2) \geq V\,,$$
$$\mathcal{A} = A_c \leq A_c + A_h\,, \tag{5.32}$$

we realize that the isoperimetric inequality (4.13) holds. Numerical investigations indicate that this inequality survives provided we took $\mathcal{V} = V$ and $\mathcal{A} = A_c$ for all admissible parameters a, b, m and q for which the metric (5.24) describes an asymptotically dS charged rotating black hole.

VI. SUMMARY

We have shown that it is possible to identify a negative thermodynamic pressure with a positive cosmological constant, allowing us to make sense of the thermodynamics of asymptotically de Sitter black holes. Both the first law of black hole thermodynamics and the corresponding Smarr relation were demonstrated to hold, and we illustrated our resulting formulae by applying them to various rotating (and charged) black holes, including

Kerr-dS black holes in all dimensions. We found that in all studied examples the thermodynamic volume in between the horizons coincided with the naive geometric one, being equal to the difference of products of cosmological and black hole horizon radii and horizon areas. We also studied the thermodynamics of the cosmological horizon and the black hole horizon separately. This allowed us to define the corresponding cosmological and black hole thermodynamic volumes. In all the examples we considered (except for the doubly-rotating Einstein-Chern-Simons-dS black hole for which we have only numerical evidence), we showed that the reverse isoperimetric inequality holds provided we take the thermodynamic volume to be that of either the black hole or the cosmological horizon; should we take the volume to be the naive geometric volume in between the horizons then the isoperimetric inequality holds. We conjecture that these relations will remain valid for any asymptotically de Sitter black holes.

An interesting case to consider for future study is the production of black holes in inflation. Without the black holes, the cosmological constant decays from the fluctuations in the scalar field that go into cosmological per-turbations, and the cosmological horizon shrinks accordingly. If black holes are produced as well, then provided $\delta A_h > 0$ the first law tells us that either, or both, of (A_c, Λ) must decrease. Perhaps there is an isoperimetric inequality that would have an interpretation in terms of such a process.

Note added. We notice some overlap of our work with recent paper [41].

VII.   ACKNOWLEDGMENTS

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